Scoring Auctions

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In the modern world, **auctions** are used to conduct a huge volume of economic transactions.

Government contracts are typically awarded by **procurement auctions**, which are also often used by firms subcontracting work or buying services and raw materials.

In OECD (2013) it is reported that the procurement of public services accounts for approximately 17% of GDP of EU countries.

The **theory of auctions** provides the necessary analytical framework to study such procurements.

Benchmark model: there is one indivisible object up for sale and there are some potential bidders.

Standard auction: the object is sold to the highest bidder.

Procurement auction: the auctioneer is the buyer and the object is sold to the lowest bidder.

The payment by each bidder depends on the type of auction used by the seller.

Huge literature around this model (see Krishna, 2010).

It may be noted that the benchmark model is really a **price-only** auction.

For example, in the traditional theory of standard procurement auctions, the auctioneer cares only about the price of the object, but not the other attributes.

However, in many procurement situations, the buyer cares about attributes other than price when evaluating the offers submitted by suppliers.

Non-monetary attributes that buyers care about include **quality**, time to completion etc.

Example- in the contract for the construction of a new aircraft, the specification of its characteristics is probably as important as its price.

Under these circumstances, procurement auctions are usually multidimensional: bidders submit bids with the relevant characteristics of the project (among which is price).

The procurement agency gives a score to each bid and makes its decisions based on these scores (**scoring auction**).

Examples of scoring auctions

The Department of Defence in USA often relies on competitive source selection to procure weapon systems.

Each individual component of a bid of the weapon system is evaluated and assigned a score, these scores are summed to yield a total score, and the firm achieving the highest score wins the contract.

Bidding for highway construction work in the United States: procurement authorities evaluate offers on the basis of price and non-price attributes.

The rule of “weighted criteria” (used in states like Delaware, Idaho, Massachusetts, Oregon, Utah, Virginia, etc.) puts a weight on each of price and quality attributes (e.g. delivery date, safety level) and evaluates each attribute individually, so that a total score of each offer is a weighted sum of sub-scores and a supplier with the highest total score wins a contract.

In a country like India where fuel costs are very high, airlines greatly value the fuel cost savings.

Airline companies in India typically purchase new aircraft after evaluating competing offers (that include price as well as various quality parameters) from big aircraft suppliers like Boeing and Airbus.

For example in 2011, after evaluating competing offers, IndiGo (a low-cost Indian airline) ordered 180 Airbus A320s from Airbus for a valuation of $15.6 billion.

The Baseline Model (Che, RAND, 1993)

A buyer solicits bids from n firms.

Each bid specifies an offer of promised quality, q and price, p, at which a fixed quantity of products with the offered level of quality q is delivered. The quantity is normalised to one.

For simplicity quality is modelled as a one-dimensional attribute.

The buyer derives utility from the contract $(p,q) \in \mathbb{R}^2_+$

$$U(p,p) = V(q) - p$$

where $V' >$ and $V'' < 0$.

A firm i upon winning, earns from a contract (q,p) profits:

$$\pi_i(p,q) = p - c(q,\theta_i)$$

where firm i's cost $c(q,\theta_i)$ is increasing in both quality q and cost parameter θ_i.

We assume $c_{qq} \geq 0$, $c_{q\theta} \geq 0$ and $c_{qq\theta} \geq 0$.

We also assume that the buyer never wishes to split the contract to more than one firm (i.e. the cost is not too convex in q).

These assumptions are satisfied by $c(q, \theta) = q\theta$.

Losing firms earn zero.

Prior to bidding each firm i learns its cost parameter θ_i as private information. The buyer and *other* firms (i.e. other than firm i) do not observe θ_i but only knows the distribution function of the cost parameter.

It is assumed that θ_i is identically and independently distributed over $[\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} < \bar{\theta}$.

Additional assumptions:

1. $c_q + \frac{F}{f} c_{q\theta}$ is non-decreasing in θ.

2. The trade always takes place (even with the highest type $\bar{\theta}$).

Let $S(p,q)$ denote a scoring rule for an offer (p,q). The rule is assumed to be publicly known to the firms at the start of bidding.

We restrict attention to quasi-linear scoring rules with the following properties:

$S(p,q) = s(q) - p$ where $s(q) - c(q,\theta)$ has a unique maximum in q for all $\theta \in [\underline{\theta},\bar{\theta}]$ and $s(.)$ is increasing in q.

The buyer awards the contract to a firm whose offer achieves the highest score. This is similar to a standard auction.

First-score auction: The winning firm's offer is finalised as the contract. This auction rule is a two-dimensional analogue of the first price auction.

In a **second-score** auction, the winning firm is required to (in the contract) to match the highest rejected score. In meeting this score, the firm is **free** to choose any quality-price combination. This auction rule is a two-dimensional analogue of the second-price auction.

An example: Let the scoring rule be $S(p, q) = 10\sqrt{q} - p$.

Suppose two firms A and B offer $(5, 16)$ and $(3, 9)$.

Note $S(5, 16) = 35$ and $S(3, 9) = 27$.

Under both auction rules firm A is declared the winner. However, the **final contract** awarded to firm A is the following:

1. $(5, 16)$ under the first-score auction.

2. Any (p, q) satisfying $S(p, q) = 27$ under the second-score auction.

Equilibria under various auction rules

Each auction rule can be viewed as a Bayesian game where each firm picks a quality-price combination (p, q) as a function of its cost parameter.

Without any loss of generality, the strategy of each firm can be **equivalently** described as picking a score and quality (S, q).

We now provide our first main result.

Lemma:
With first-score and second-score auctions, quality is chosen at $q_s(\theta)$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$ where

$$q_s(\theta) = \arg\max_q s(q) - c(q,\theta).$$

A simple intuition behind the result is that in equilibrium the firm tries to maximise $p - c(q,\theta)$ given a score level $\bar{s}$.

Essentially, the previous lemma reduces the two dimensional auction to a single dimensional problem.

Let in equilibrium each firm choose (p, q) as a function of its type θ. That is a firm chooses $(p(\theta), q(\theta))$. In equilibrium $q(\theta) = q_s(\theta)$.

It can be shown that there is a Bayesian-Nash equilibrium where the score $S(p(\theta), q_s(\theta))$ chosen is strictly decreasing in θ.

Let

$$
\begin{aligned}
S_0(\theta) &= \max(s(q) - c(q,\theta)) \\
&= s(q_s(\theta)) - c(q_s(\theta),\theta).
\end{aligned}
$$

From the envelope theorem $S_0(.)$ is strictly decreasing.

Consider the following change of variables:

$$
\begin{aligned}
v &\equiv S_0(\theta) \equiv s(q_s(\theta)) - c(q_s(\theta),\theta) \\
H(v) &\equiv 1 - F(S_0^{-1}(v)) \\
b(\theta) &\equiv S(q_s(\theta),p(\theta)) \equiv s(q_s(\theta)) - p(\theta)
\end{aligned}
$$

Note that the objective function facing each firm in the **first-score** auction is the following:

$$\pi(q_s(\theta), p \mid \theta) = [p - c(q_s(\theta), \theta)]\left(\text{prob. win}\right)$$

Firm 1 (say) will win the contract iff it has the highest score. That is, if

$$b(\theta_1) > \max_{j \neq 1}\{b(\theta_j)\}$$

$$\Leftrightarrow \theta_1 < \min_{j \neq 1}\{\theta_j\}$$

(since $b(.)$ is strictly decreasing)

$$\text{prob. win} = \text{prob.}\left(\theta_1 < \min_{j \neq 1}\{\theta_j\}\right)$$

$$= (1 - F(\theta_1))^{n-1}$$

Note $p - c(q_s(\theta), \theta) = v - b(\theta)$

Hence we have

$$\pi(q_s(\theta), p \mid \theta) = [v - b(\theta)](1 - F(\theta))^{n-1}$$
$$= [v - b(\theta)]H(v)$$

Symmetric equilibrium of a **first-score auction**:

$$q_s(\theta) = \arg\max_q s(q) - c(q, \theta)$$

$$p_s(\theta) = c(q_s(\theta), \theta) + \int_\theta^{\bar{\theta}} c_\theta(q(t), t)\left[\frac{1 - F(t)}{1 - F(\theta)}\right]^{n-1}$$

The **second-score** auction has a dominant strategy equilibrium:

$$q_s(\theta) = \arg\max_q s(q) - c(q, \theta)$$

$$p_s(\theta) = c(q_s(\theta), \theta).$$

Score equivalence:

With the scoring rule ($S(q,p) = V(q) - p$) first-score and second-score auctions yield the same expected utility to the buyer, equal to

$$E\left\{\begin{array}{c} V(q^*(\theta_1)) - c(q^*(\theta_1),\theta_1) \\ -\frac{F(\theta_1)}{f(\theta_1)} c_\theta(q^*(\theta_1),\theta_1) \end{array}\right\}$$

where

$$q^*(\theta_1) = \arg\max_q V(q) - c(q,\theta_1)$$

- Two-dimensional analogue of the **Revenue Equivalence Theorem** in the benchmark model.

What happens when quality and types are **multidimensional**?

Can we reduce the strategic environment to one dimensional private information?

If so, under what conditions can this be achieved?

To answer the above questions, we need a slightly modified model (following Asker and Cantillon, RAND, 2008).

Consider a buyer seeking to procure an indivisible good for which there are n potential suppliers. The good is characterized by its price, p, and $m > 1$ non-monetary attributes, $\mathbf{Q} \in \mathbb{R}_+^m$.

The buyer values the good $(p, \mathbf{Q})$ at $v(\mathbf{Q}) - p$.

Supplier i's profit from selling good $(p, \mathbf{Q})$ is given by $p - c(\mathbf{Q}, \theta_i)$, where $\theta_i \in \mathbb{R}^k$, $k > 1$, is supplier i's type. We allow suppliers to be flexible with respect to the level of non-monetary attributes they can supply.

Preferences are common knowledge among suppliers and the buyer, with the exception of suppliers' types, $\theta_i,\ i = 1, \ldots n$, which are privately observed.

Types are independently distributed according to the continuous joint density function f_i with support on a bounded and convex subset of $\mathbb{R}^k$ with a non-empty interior Θ_i.

A scoring rule is a function $S : \mathbb{R}_+^{m+1} \longrightarrow \mathbb{R}$: $(p, \mathbf{Q}) \rightarrow S(p, \mathbf{Q})$ that associates a score to any potential contract and represents a continuous preference relation over contract characteristics $(p, \mathbf{Q})$.

The **outcome** of the scoring auction is a probability of winning the contract, x_i, a score to fulfill when the supplier wins the contract, t_i^w, and a payment to the buyer in case he does not, t_i^l.

In a first-score auction, the winner must deliver a contract that generates the value of his winning score, that is, $t_i^w = S(p, \mathbf{Q}_i)$ and $t_i^l = 0$.

In a second-score auction, the winner must deliver a contract that generates a score equal to the score of the second-best offer received and $t_i^l = 0$.

Consider supplier i with type θ_i, who has won the contract with a score to fulfill t_i^w.

Supplier i will choose characteristics $(p, \mathbf{Q})$ that maximize his profit, that is,

$$\max_{(p,\mathbf{Q})} \{p - c(\mathbf{Q}, \theta_i)\} \text{ s.t. } s(\mathbf{Q}) - p = t_i^w$$

Substituting for p into the objective function yields

$$\max_{\mathbf{Q}} \{s(\mathbf{Q}) - c(\mathbf{Q}, \theta_i) - t_i^w\}$$

An important feature of the above is that the optimal $\mathbf{Q}$ is independent of t_i^w.

Now define

$$k(\theta_i) = \max_{\mathbf{Q}}\{s(\mathbf{Q}) - c(\mathbf{Q}, \theta_i)\}$$

We shall call $k(\theta_i)$ supplier i's **pseudotype**. Bidders' pseudotypes are well defined as soon as the scoring rule is given.

The set of supplier i's possible pseudotypes is an interval in $\mathbb{R}$. The density of pseudotypes inherits the smooth properties of f_i.

With this definition, supplier i's expected profit is given by

$$x_i(k(\theta_i) - t_i^w) - (1 - x_i)t_i^l$$

In the above supplier i's preference over contracts of the type (x_i, t_i^w, t_i^l) is **entirely** captured by his pseudotype.

Note: *Only quasi-linear* scoring rules have the above property when private information is multidimensional.

Let

$$s_i = x_i t_i^w + (1 - x_i) t_i^l.$$

Given suppliers' risk neutrality and the linearity of the scoring rule, there is no loss in defining the outcome of a scoring auction as the pair (x_i, s_i), rather than (x_i, t_i^w, t_i^l).

Suppliers' expected payoff is thus given by

$$x_i k(\theta_i) - s_i.$$

The outcome function of a scoring auction is a vector of probabilities of winning $(x_1, x_2 \ldots, x_n)$ and scores to fulfill by each supplier, $(s_1, s_2 \ldots, s_n)$.

The arguments in these functions are the bids submitted by all suppliers, $\{(p_i, \mathbf{Q}_i)\}_{i=1}^{n}$.

Define two equilibria as **typewise outcome equivalent** if they generate the same distribution of outcomes $(x_1, x_2 \ldots, x_n)$ and $(s_1, s_2 \ldots, s_n)$, **conditional on types** in $\Theta_1 \times \Theta_2 \ldots \Theta_n$.

Proposition:

Every equilibrium in the scoring auction is typewise outcome equivalent to an equilibrium in the scoring auction where suppliers are constrained to bid only on the basis of their pseudotypes, and vice versa.

- The above Proposition ensures that there is no loss of generality in concentrating on pseudotypes when deriving the equilibrium in the scoring auction, even if the scoring rule does not correspond to the buyer's true preference.

- Note that the above Proposition does not rule out equilibria where different types submit different $(p, \mathbf{Q})$ bids- but given that they yield the same score and the same probability of winning at equilibrium, they are payoff irrelevant.

While the above comments might not be totally surprising when types are one-dimensional, this result is not trivial for environments where types are multidimensional.

This property (that there is no loss of generality in concentrating on *pseudotypes* when deriving the equilibrium in the scoring auction) is a consequence of the combination of the following:

1. **quasi-linear** scoring rule

2. the single dimensionality of the allocation decision

3. the independence of types across bidders.

- We cannot reduce the strategic environment to one dimensional private information if any of the above conditions does not hold.

The equilibrium in quasi-linear scoring auctions with independent types inherits the properties of the equilibrium in the related single-object auction where

1. bidders are risk neutral

2. their (private) valuations for the object correspond to the pseudotype k in the original scoring auction and are distributed accordingly

3. the highest bidder wins

4. the payment rule is determined as in the scoring auction, with bidders' scores being replaced by bidders' bids.

The above suggests the following simple algorithm for deriving equilibria in scoring auctions:

1. Given the scoring rule, derive the distribution of pseudotypes, $G_i(k)$.

2. Solve for the equilibrium in the related SIPV (benchmark) auction where valuations are distributed according to $G_i(k)$.

3. The equilibrium bid in the scoring auction is any $(p, \mathbf{Q})$ such that $S(p, \mathbf{Q}) = b_i(k)$.

4. The actual $(p, \mathbf{Q})$ delivered are easy to derive given $b_i(k)$ and the solution to
$$\max_{\mathbf{Q}} \{s(\mathbf{Q}) - c(\mathbf{Q}, \theta_i) - t_i^w\}.$$

Non-quasilinear scoring rules

Most papers on scoring auctions, except a very few recent ones, have used quasilinear scoring rules. That is, $S(p,q) = \phi(q) - p$.

What about the case where the scoring rules are **non-quasilinear**? Why should we care for such scoring rules?

1. equilibrium properties of scoring auctions with general non-quasilinear scoring rules have not been fully worked out

2. non-quasilinear scoring rules are often used in real life.

Examples

For highway construction projects, states like Alaska, Colorado, Florida, Michigan, North Carolina, and South Dakota use **quality-over-price** ratio rules, in which the score is computed based on the **quality divided by price** (i.e. $S(p,q) = \frac{q}{p}$).

The above scoring rule is extensively used in Japan and also in Australia.

Ministry of Land, Infrastructure and Transportation in Japan allocates most of the public construction project contracts through scoring auctions based on quality-over-price ratio rules

In its Guide to Greener Purchasing, the OECD (2000, p.12) writes that the objective of procurement rules in member countries is "to achieve a transparent and verifiable best price/quality ratio for any given product or service." Quality-price ratios are thus used explicitly for assessing bids for procurement purposes by many governments.

Some governments in EU countries use the scoring auction in which the score is the sum of the price and quality measurements but the score is nonlinear in the price bid (see Nakabayashi et al, 2014).

However, very few papers in the literature have dealt with general **non-quasilinear** scoring rules.

1. Hanazono, Nakabayashi and Tsuruoka (2015) is the only paper till date that deals with **general** non-quasilinear scoring rules.

2. Hanazono (2010, *Economic Science*, in Japanese) provides an example with a **specific** non-quasilinear scoring rule and a specific cost function.

3. Wang and Liu (2014, Economics Letters) analyses equilibrium properties of first-score auctions with another **specific** non-quasilinear scoring rule.

Note the following for all the above papers.

1. The **explicit solutions** for the equilibrium strategies are **not generally obtained**.

2. The choice of ‘quality’ is **endogenous** in the ‘score’ under the general scoring function.

3. Moreover, the comparison of expected scores (in Hanazono el at, 2015) is based on properties of *induced utility whose arguments are implicitly defined.*

Questions (**Dastidar**, **2015**)

1. *Can we get* ***explicit solutions*** *for equilibrium strategies with general non-quasilinear scoring rules?*

2. Can we provide a **complete characterisation** (price, quality, score) of such equilibria?

3. *Also, can we get ranking of the two auction formats (first-score and second-score) in terms of expected scores by directly using curvature properties of the scoring rule and the distribution function of types?*

4. *If so, under what conditions can the above be achieved?*

Answer:

We show that all the above can be done if the cost function is ***additively separable*** *in quality and type*.

1. Our computations provide a much simpler way to derive equilibria in scoring auctions without any endogeneity problems. We get ***explicit solutions***.

2. We provide a **complete characterisation** of such equilibria and **ranking** of quality, price and the expected scores.

3. This stands in contrast to the results derived in Hanazono et al (2015) and Wang and Liu (Economics Letters, 2014).

The Model

A buyer solicits bids from n firms.

Each bid, $(p, q) \in \mathbb{R}^2_{++}$, specifies an offer of promised quality, q and price, p, at which a fixed quantity of products with the offered level of quality q is delivered.

The quantity is normalized to one. For simplicity quality is modelled as a one-dimensional attribute.

The buyer awards the contract to a firm whose offer achieves the **highest score**.

Scoring rule : $S(p,q) : \mathbb{R}^2_{++} \to \mathbb{R}$

Assumption 1

$S(.)$ is strictly decreasing in p and strictly increasing in q. That is, $S_p < 0$ and $S_q > 0$. We assume that the partial derivatives S_p, S_q, S_{pp}, S_{pq}, S_{qq} exist and they are continuous in all $(p,q) \in \mathbb{R}^2_{++}$.

A scoring rule is **quasilinear** if it can be expressed as $\phi(q) - p$ or any monotonic increasing function thereof. For quasilinear rules we have $S_{pp} = 0$ and $S_{pq} = 0$.

For **non-quasi-linear** rules we must have at least one of the following: $S_{pp} \neq 0$ or $S_{pq} \neq 0$.

The cost to the supplier is $C(q,x)$ where x is the type.

Assumption 2
We assume $C_q > 0$, $C_{qq} \geq 0$ and $C_x > 0$.

Prior to bidding each firm i learns its cost parameter x_i as private information.

The buyer and *other* firms (i.e. other than firm i) do not observe x_i but only knows the distribution function of the cost parameter.

It is assumed that x_is are identically and independently distributed over $[\underline{x}, \bar{x}]$ where $0 \leq \underline{x} < \bar{x}$.

If supplier i wins the contract, its payoff is $p - C(q, x_i)$.

Assumption 3

Cost is additively separable in quality and type.

That is, $C(q,x) = c(q) + \alpha(x)$ where $c'(.) > 0$, $c''(.) \geq 0$, $\alpha(\underline{x}) > 0$ and $\alpha'(.) > 0$.

Define $\theta_i = \alpha(x_i)$.

Let $\underline{\theta} = \alpha(\underline{x})$ and let $\bar{\theta} = \alpha(\bar{x})$. Clearly, $0 \leq \underline{\theta} < \bar{\theta}$.

Since x_is are identically and independently distributed over $[\underline{x},\bar{x}]$, so are the θ_is over $[\underline{\theta},\bar{\theta}]$.

Let the distribution function of θ_i be $F(.)$ and the density function be $f(.)$.

Note that $f(\theta) \geq 0 \; \forall\theta \in [\underline{\theta}, \bar{\theta}]$.

We can now write the cost for supplier as $C(q, \theta_i) = c(q) + \theta_i$, where θ_i is the type of supplier i.

Assumption 4

For all $(p,q) \in \mathbb{R}^2_{++}$

$$-\frac{(S_q)^2}{S_p} S_{pp} + 2S_q S_{pq} - S_p S_{qq} - (S_p)^2 c''(.) < 0$$

- It may also be noted that when $c''(q) > 0$ then both for the quasilinear rule

$$S(p,q) = q - p$$

and for the non-quasilinear rule

$$S(p,q) = \frac{q}{p}$$

the above assumption is always satisfied.

The following may be noted:

1. Our cost, $C(q, \theta_i) = c(q) + \theta_i$, can be interpreted in the following way. $c(q)$ is the variable cost and θ_i is the fixed cost of firm i. This means, the variable costs are same across firms but the fixed costs are private information.

2. θ_i can be interpreted to be the inverse of managerial/engineering efficiency which is private information to the firm.

3. Higher is θ_i, lower is the managerial/engineering efficiency, and consequently, higher will be the cost.

4. The assumption (cost is additively separable in quality and type) is consistent with the set of assumptions in Hanazono et al (2015) and Asker and Cantillon (Rand, 2008).

5. Additive separability implies $C_{q\theta}(.) = 0$. This is different from Che (Rand, 1993), Branco (Rand, 1997) and Nishimura (2015).

Proposition 1

In a *first-score* auction there is a symmetric equilibrium where a supplier with type θ chooses $(p^I(\theta), q^I(\theta))$. Such $p^I(.)$ and $q^I(.)$ are obtained by solving the following equations:

$$-\frac{S_q(.)}{S_p(.)} = c'(.)$$

$$p - c(q) = \theta + \gamma(\theta)$$

where

$$\gamma(\theta) = \frac{1}{(1 - F(\theta))^{n-1}} \int_{\theta}^{\bar{\theta}} (1 - F(t))^{n-1} dt$$

Proposition 2

In a *second-score* auction there is a weakly dominant strategy equilibrium where a supplier with type θ chooses $(p^{II}(\theta), q^{II}(\theta))$. Such $p^{II}(.)$ and $q^{II}(.)$ are obtained by solving the following equations:

$$- \frac{S_q(.)}{S_p(.)} = c'(.)$$

$$p - c(q) = \theta$$

- Our equilibrium is similar to Che (Rand, 1993).

- In Hanazono et al (2015) the scoring rule is non-quasilinear but the equilibrium strategies are only derived implicitly. Same is true for Wang and Liu (Eco. Let., 2014), where a specific scoring rule is considered.

- In our case, the cost function is additively separable in quality and type and we get explicit solutions for equilibrium strategies for both kinds of scoring rules: quasilinear and non-quasilinear.

- Additive separability of the cost function makes equilibrium computations very simple. This stands in contrast to Hanazono et al (2015) and Wang and Liu (Eco. Let., 2014).

- Moreover, our assumptions are also milder and are satisfied a by a large class of scoring rules.

- When the scoring rule is **quasilinear** $S_p(.)$ is a constant and S_q is independent of p (since $S_{pp} = S_{qp} = 0$). Note that in any auction the equation $-\frac{S_q(.)}{S_p(.)} = c'(.)$ is satisfied. This means the quality, q, is constant and same for the two auctions.

We illustrate the above two propositions in two examples given below.

Example 1 (non-quasilinear scoring rule)

Let $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. Let θ be uniformly distributed over $[1,2]$. Let $n = 2$.

Equilibrium

First-score auction:

$$p^I(\theta) = 2 + \theta, \;\; q^I(\theta) = \sqrt{2 + \theta} \;\; \forall\theta \in [1,2].$$

Second-score auction:

$$p^{II}(\theta) = 2\theta, \; q^{II}(\theta) = \sqrt{2\theta} \;\; \forall\theta \in [1,2].$$

Example 2 (quasilinear scoring rule)

Let $S(p,q) = q - p$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. Let θ be uniformly distributed over $[1,2]$ and $n = 2$.

Equilibrium

First-score auction

$$p^I(\theta) = \frac{3}{2} + \frac{1}{2}\theta, \ \ q^I(\theta) = 1 \ \forall\theta \in [1,2].$$

Second-score auction

$$p^{II}(\theta) = \frac{1}{2} + \theta, \ q^{II}(\theta) = 1 \ \forall\theta \in [1,2].$$

We define the following:

$$A(p,q) = -\frac{S_q(p,q)}{S_p(p,q)} S_{pp}(p,q) + S_{qp}(p,q)$$

$$B(p,q) = -\frac{S_q(p,q)}{S_p(p,q)} S_{pq}(p,q) + S_p(p,q) c''(q) + S_{qq}(p,q)$$

Equilibrium Characterisation

Lemma 1

$p^I(\bar{\theta}) = p^{II}(\bar{\theta})$ and $q^I(\bar{\theta}) = q^{II}(\bar{\theta})$.

- A firm with the highest type ($\bar{\theta}$) quotes the same price and quality across first-score and second-score auctions (lemma 1).
- Consequently, a firm with type $\bar{\theta}$ also quotes the **same score** in both auctions. This is true regardless of the fact whether the scoring rule is quasilinear or not.

The following lemma links the sign of $A(p,q)$ and $B(p,q)$.

Lemma 2
Suppose $A(p,q) \neq 0 \ \forall (p,q) \in \mathbb{R}^2_{++}$.

$$B(p,q) \geq 0 \Rightarrow A(p,q) < 0.$$

We now proceed to consider scoring rules that are **non-quasilinear**.

For such rules we must have at least one of the following: $S_{pp} \neq 0$, $S_{pq} \neq 0$.

Let $S^I(\theta) = S(p^I(\theta), q^I(\theta))$

and $S^{II}(\theta) = S^{II}(p^{II}(\theta), q^{II}(\theta))$.

In the first-score and second-score auctions the equilibrium scores quoted by a firm with type θ is $S^I(\theta)$ and $S^{II}(\theta)$ respectively.

Proposition 3
If $A(p,q) \neq 0 \;\; \forall (p,q) \in \mathbb{R}^2_{++}$ then $S^I(\theta) < S^{II}(\theta) \; \forall \theta \in [\underline{\theta}, \bar{\theta})$.

Also, $\frac{d}{d\theta} S^I(\theta), \frac{d}{d\theta} S^{II}(\theta) < 0 \; \forall \theta \in (\underline{\theta}, \bar{\theta})$.

- The equilibrium score quoted by any type $\theta \in [\underline{\theta}, \bar{\theta})$ is strictly higher in the second-score auction as compared to the equilibrium score in first score-auction.

- This is analogous to the standard benchmark model where for any particular type, the bid in the second-price auction is always higher than the bid in the first-price auction.

- Proposition 3 also shows that equilibrium scores are decreasing in type, θ. This means the winner in any auction is the firm with the lowest type (least cost).

- That is, the symmetric equilibria are always efficient.

Proposition 4

(i) If $A(p,q) > 0 \;\; \forall (p,q) \in \mathbb{R}^2_{++}$ then $q^I(\theta) > q^{II}(\theta) \; \forall \theta \in [\underline{\theta}, \bar{\theta})$. Also, $\frac{dq^I(\theta)}{d\theta}, \frac{dq^{II}(\theta)}{d\theta} > 0 \;\; \forall \theta \in (\underline{\theta}, \bar{\theta})$.

(ii) If $A(p,q) < 0 \;\; \forall (p,q) \in \mathbb{R}^2_{++}$ then $q^I(\theta) < q^{II}(\theta) \;\; \forall \theta \in [\underline{\theta}, \bar{\theta})$. Also, $\frac{dq^I(\theta)}{d\theta}, \frac{dq^{II}(\theta)}{d\theta} < 0 \;\; \forall \theta \in (\underline{\theta}, \bar{\theta})$.

(iii) If $A(p,q) = 0 \;\; \forall (p,q) \in \mathbb{R}^2_{++}$ then $q^I(\theta) = q^{II}(\theta) \;\; \forall \theta \in [\underline{\theta}, \bar{\theta})$. Also, $\frac{dq^I(\theta)}{d\theta}, \frac{dq^{II}(\theta)}{d\theta} = 0 \;\; \forall \theta \in (\underline{\theta}, \bar{\theta})$.

Proposition 5

Suppose $A(p,q) \neq 0 \ \forall (p,q) \in \mathbb{R}^2_{++}$.

(i) If $B(p,q) < 0 \ \forall (p,q) \in \mathbb{R}^2_{++}$ then $p^I(\theta) > p^{II}(\theta) \ \forall \theta \in [\underline{\theta}, \bar{\theta})$. Also, $\frac{dp^I(\theta)}{d\theta}, \ \frac{dp^{II}(\theta)}{d\theta} > 0 \ \forall \theta \in (\underline{\theta}, \bar{\theta})$.

(ii) If $B(p,q) > 0 \ \forall (p,q) \in \mathbb{R}^2_{++}$ then $p^I(\theta) < p^{II}(\theta) \ \forall \theta \in [\underline{\theta}, \bar{\theta})$. Also, $\frac{dp^I(\theta)}{d\theta}, \ \frac{dp^{II}(\theta)}{d\theta} < 0 \ \forall \theta \in (\underline{\theta}, \bar{\theta})$.

(iii) If $B(p,q) = 0 \ \forall (p,q) \in \mathbb{R}^2_{++}$ then $p^I(\theta) = p^{II}(\theta) \ \forall \theta \in [\underline{\theta}, \bar{\theta}]$. Also, $\frac{dp^I(\theta)}{d\theta}, \ \frac{dp^{II}(\theta)}{d\theta} = 0 \ \forall \theta \in (\underline{\theta}, \bar{\theta})$.

From propositions 4 and 5 the following emerge:

- Sign of the term $A(p,q)$ plays a crucial role in determining for characterisation of equilibrium quality quoted in any auction.
- Sign of the term $B(p,q)$ plays a crucial role in determining for characterisation of equilibrium price quoted in any auction.
- Note that lemma 2 links the sign of $A(p,q)$ and $B(p,q)$.

From lemma 2 we get

$A(p,q) > 0 \Rightarrow B(p,q) < 0$.

The above in combination with propositions 4 and 5 means that $A(p,q) > 0$ implies $q^I(\theta) > q^{II}(\theta)$ and $p^I(\theta) > p^{II}(\theta)$. Also $\frac{dq^I(\theta)}{d\theta}, \frac{dq^{II}(\theta)}{d\theta} > 0$ and $\frac{dp^I(\theta)}{d\theta}, \frac{dp^{II}(\theta)}{d\theta} > 0$.

Similarly, $B(p,q) \geq 0 \Rightarrow A(p,q) < 0$ and we have $q^I(\theta) < q^{II}(\theta)$ and $p^I(\theta) < p^{II}(\theta)$. Also $\frac{dq^I(\theta)}{d\theta}, \frac{dq^{II}(\theta)}{d\theta} < 0$ and $\frac{dp^I(\theta)}{d\theta}, \frac{dp^{II}(\theta)}{d\theta} < 0$.

We now provide a few examples to illustrate propositions 4 and 5.

The point is to show that scoring rules and cost functions exist that satisfy all our assumptions and the conditions of propositions 4 and 5.

We first consider conditions mentioned in proposition 4.

1. $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. In this example $A(.) > 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.

2. $S(p,q) = 10q - p^2$ and $C(q,\theta) = q + \theta$. In this example $A(.) < 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.

3. $S(p,q) = e^{q-p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. In this example $A(.) = 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.

We now consider conditions mentioned in proposition 5.

1. $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. In this example $B(.) < 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.

2. $S(p,q) = e^{q-p} - p$ and $C(q,\theta) = \frac{1}{2}q + \theta$. In this example $B(.) > 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.

3. $S(p,q) = 10q - p^2$ and $C(q,\theta) = q + \theta$. In this example $B(.) = 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.

We now proceed to discuss the impact of increase in n (the number of bidders) on equilibrium quality and price in both auctions.

For any given θ, let $q^{I}(n;\theta)$ and $q^{II}(n;\theta)$ be the quality quoted in first-score and second-score auctions respectively when the number of bidders is n.

Similarly, for any given θ, let $p^{I}(n;\theta)$ and $p^{II}(n;\theta)$ be the price quoted in first-score and second-score auctions respectively when the number of bidders is n.

Proposition 6

For all $n > m$

(i) $q^{II}(n;\theta) = q^{II}(m;\theta)$.

(ii) If $A(p,q) > 0 \ \ \forall (p,q) \in \mathbb{R}^2_{++}$ then $q^I(n;\theta) < q^I(m;\theta)$.

(iii) If $A(p,q) < 0 \ \ \forall (p,q) \in \mathbb{R}^2_{++}$ then $q^I(n;\theta) > q^I(m;\theta)$.

Proposition 7

Suppose $A(p,q) \neq 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$. Then for all $n > m$

(i) $p^{II}(n;\theta) = p^{II}(m;\theta)$.

(ii) If $B(p,q) = 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$ then $p^I(n;\theta) = p^I(m;\theta)$.

(iii) If $B(p,q) > 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$ then $p^I(n;\theta) > p^I(m;\theta)$.

(iv) If $B(p,q) < 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$ then $p^I(n;\theta) < p^I(m;\theta)$.

The next proposition explores how the equilibrium score quoted changes with an increase in the number of bidders.

Let $S^{I}(n;\theta) = S(p^{I}(n;\theta), q^{I}(n;\theta))$

and $S^{II}(n;\theta) = S(p^{II}(n;\theta), q^{II}(n;\theta))$.

Proposition 8
(i) For all $n > m$, $S^{II}(n;\theta) = S^{II}(m;\theta)$.

(ii) For all $n > m$, $S^{I}(n;\theta) > S^{I}(m;\theta)$.

- In the second-score auction the quality and price quoted in equilibrium are independent of the number of bidders. Consequently, the score quoted in equilibrium is invariant with respect to the number of bidders.

- This is similar to the second-price auction in the benchmark model, where, regardless of the number of bidders, all bidders bid their valuations.

- In the first-score auction as the competition intensifies (n increases) the score quoted by any type increases. This is in line with the conventional wisdom which suggests that any increase in competition should induce a bidder with type θ to quote a higher score.

- This is also similar to the first-price auction in the benchmark model where bids increase with the number of bidders.

Expected Scores:

Let $F_1(.)$ and $f_1(.)$ be the distribution and density function of the lowest order statistic.

Let $F_2(.)$ and $f_2(.)$ be the distribution and density function of the second lowest order statistic.

$$
\begin{aligned}
F_1(x) &= 1-(1-F(x))^n \\
F_2(x) &= 1-(1-F(x))^n - nF(x)(1-F(x))^{n-1} \\
f_1(x) &= n(1-F(x))^{n-1}f(x) \\
f_2(x) &= n(n-1)F(x)(1-F(x))^{n-2}f(x)
\end{aligned}
$$

Lemma 3

(i) In a first-score auction the expected score is as follows:

$$\Sigma^I = \int_{\underline{\theta}}^{\bar{\theta}} S^I(\theta) f_1(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} S(p^I(\bar{\theta}), q^I(\bar{\theta})) f_1(\theta) d\theta$$

$$= S(p^I(\bar{\theta}), q^I(\bar{\theta}))$$

$$- \int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)(1 + \gamma'(\theta)) S_p(p^I(\theta), q^I(\theta)) d\theta$$

where

$$\gamma(\theta) = \frac{1}{(1 - F(\theta))^{n-1}} \int_{\theta}^{\bar{\theta}} (1 - F(t))^{n-1} dt$$

(ii) In a second-score auction the expected score is as follows:

$$\Sigma^{II} = \int_{\underline{\theta}}^{\bar{\theta}} S^{II}(\theta) f_2(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} S(p^{II}(\bar{\theta}), q^{II}(\bar{\theta})) f_2(\theta) d\theta$$

$$= S(p^{II}(\bar{\theta}), q^{II}(\bar{\theta}))$$

$$- \int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta) S_p(p^{II}(\theta), q^{II}(\theta)) d\theta$$

Lemma 4

$$\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)(1+\gamma'(\theta))d\theta$$
$$= \int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta)d\theta$$

where

$$\gamma(\theta) = \frac{1}{(1-F(\theta))^{n-1}} \int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1} dt$$

From lemma 1 we know $p^I(\bar{\theta}) = p^{II}(\bar{\theta})$ and $q^I(\bar{\theta}) = q^{II}(\bar{\theta})$.

This means

$$S(p^I(\bar{\theta}), q^I(\bar{\theta})) = S(p^{II}(\bar{\theta}), q^{II}(\bar{\theta})).$$

Using this and lemma 3 one clearly gets that to compare Σ^I and Σ^{II} we need to compare the following terms:

$$\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)(1 + \gamma'(\theta))[-S_p(p^I(\theta), q^I(\theta))]d\theta$$

and $\int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta)[-S_p(p^{II}(\theta), q^{II}(\theta))]d\theta.$

Note that if the scoring rule is **quasilinear** (i.e. $S(p,q) = \phi(q) - p$) then $S_p = -1$.

Hence, from lemmas 3 and 4 the next result follows.

Proposition 9
If the scoring rule is quasilinear then
$\Sigma^I = \Sigma^{II}$.

- The above result is well known. For scoring auctions this is the analogue of *revenue equivalence theorem* of the canonical model.

We now proceed to provide our main results on expected scores when the scoring rules are **non-quasilinear**.

We show that such results will depend on the curvature properties of the scoring rule and the properties of the distribution function of types.

We first show the possibility of equivalence of expected scores even with non-quasilinear scoring rules.

Proposition 10
If $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(.) \neq 0$ and $S_{pp}\frac{B(.)}{A(.)} - S_{pq} = 0$ then $\Sigma^I = \Sigma^{II}$.

We illustrate proposition 10 with a couple of examples.

In one example $S_{pq} = 0$ and in the other example $S_{pq} \neq 0$.

Example 3:
Let $S(p,q) = 10q - p^2$, $C(q,\theta) = q + \theta$ and θ is uniformly distributed over $[1,2]$. The scoring rule is non-quasilinear and satisfies all our assumptions. Here it can be easily shown that $\Sigma^I = \Sigma^{II} = \frac{25}{3}$.

Example 4:
Let $S(p,q) = e^{q-p} - p$, $C(q,\theta) = \frac{1}{2}q + \theta$ and θ is uniformly distributed over $[\frac{1}{4}, \frac{1}{2}]$. The scoring rule is non-quasilinear and satisfies all our assumptions. Here we have $\Sigma^I = \Sigma^{II} = \frac{1}{6}$.

From proposition 10 we get that

$$\Sigma^I \neq \Sigma^{II} \Rightarrow S_{pp}\frac{B(.)}{A(.)} - S_{pq} \neq 0 \text{ for some } (p,q) \in \mathbb{R}^2_{++}$$

Now suppose the scoring rule is such that $S_{pp}\frac{B(.)}{A(.)} - S_{pq} \neq 0$ for some $(p,q) \in \mathbb{R}^2_{++}$.

We now show that a restriction on the distribution function of types ensures $\Sigma^I < \Sigma^{II}$.

Proposition 11

Suppose the scoring rule, $S(.)$, is non-quasilinear and $S_{pp}\frac{B(.)}{A(.)} - S_{pq} \neq 0$ for some $(p,q) \in \mathbb{R}^2_{++}$. If $f'(\theta) \leq 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$ and $f(\bar{\theta})$ is large enough then $\Sigma^I < \Sigma^{II}$.

- *Proposition 11 is interesting as it demonstrates the need to put restrictions on the distribution function of types to get a ranking of expected scores. This stands in sharp contrast to the other papers in the literature.*

- It may be noted that most non-quasilinear scoring rules, including the quality over price ratio, satisfy the restriction $S_{pp}\frac{B(.)}{A(.)} - S_{pq} \neq 0$. Also, the restriction, $f'(\theta) \leq 0$, is satisfied by many distribution functions (including the uniform distribution).

- As such, the expected scores will be strictly higher with second-score auctions for most scoring rules and many distribution functions.

- This has interesting policy implications as well. In real life second-score auctions are never used. Our result suggests that in a large number of cases an auctioneer will be better off using second-score auctions than using first-score auctions.

We now illustrate this result with two examples. We take the ‘quality over price’ scoring rule and the same quadratic cost function in both examples.

Note that the restriction $S_{pp}\frac{B(.)}{A(.)} - S_{pq} \neq 0$ is satisfied for this scoring rule and cost function.

The distribution function of types are different in the two examples.

Proposition 11 demonstrates that when $S_{pp}\frac{B(.)}{A(.)} - S_{pq} \neq 0$ for some $(p,q) \in \mathbb{R}^2_{++}$ then $\Sigma^I \geq \Sigma^{II}$ implies that at least one of the following is true: (i) $f'(\theta) > 0$ or (ii) $f(\bar{\theta})$ is not large enough.

In example 5 we take a uniform distribution, where $f'(\theta) = 0$ and show that $\Sigma^I < \Sigma^{II}$.

In example 6, we take a different distribution function where $f'(.) > 0$ and we get $\Sigma^I > \Sigma^{II}$.

Example 5

Let $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$.

Suppose θ be uniformly distributed over $[1,2]$ and $n = 2$.

For this distribution we have

$$f_1(\theta) = 2(2-\theta) \text{ and } f_2(\theta) = 2(\theta - 1)$$

First-score auction:

price: $p^I(\theta) = 2 + \theta$

quality: $q^I(\theta) = \sqrt{2 + \theta}$

score: $s^I(\theta) = \frac{q^I(\theta)}{p^I(\theta)} = \frac{1}{\sqrt{2+\theta}}$

Expected score:

$\Sigma^I = \int_1^2 s^I(\theta) f_1(\theta) d\theta = 0.54872$

Second-score auction:

price: $p^{II}(\theta) = 2\theta$

quality: $q^{II}(\theta) = \sqrt{2\theta}$

score: $s^{II}(\theta) = \frac{q^{II}(\theta)}{p^{II}(\theta)} = \frac{1}{\sqrt{2\theta}}$

Expected score:

$\Sigma^{II} = \int_1^2 s^{II}(\theta) f_2(\theta) d\theta = 0.55228$

Example 6

Let $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$.

Now suppose $n = 2$ and θ is distributed over $\left[1.2,\ 1.203731\right]$ with density $f(x) = 500x^3 - 600$ and distribution function $F(x) = 125x^4 - 600x + \frac{2304}{5}$.

For this distribution we have
$f_1 = 2\left(-125x^4 + 600x - \dfrac{2299}{5}\right)(500x^3 - 600)$ and
$f_2 = 2\left(125x^4 - 600x + \dfrac{2304}{5}\right)(500x^3 - 600)$

First-score auction:

price: $p^I(\theta) = 2\left(\theta + \frac{25\theta^5 - 300\theta^2 + \frac{45980}{10} - \frac{18197}{100}}{-125\theta^4 + 600\theta - \frac{2299}{5}}\right)$

quality: $q^I(\theta) = \sqrt{2\left(\theta + \frac{25\theta^5 - 300\theta^2 + \frac{45980}{10} - \frac{18197}{100}}{-125\theta^4 + 600\theta - \frac{2299}{5}}\right)}$

score:

$$s^I(\theta) = \frac{q^I(\theta)}{p^I(\theta)} = \frac{1}{\sqrt{2\left(\theta + \frac{25\theta^5 - 300\theta^2 + \frac{45980}{10} - \frac{18197}{100}}{-125\theta^4 + 600\theta - \frac{2299}{5}}\right)}}$$

Expected score:

$$\Sigma^I = \int_{1.2}^{1.203731} s^I(\theta) f_1(\theta) d\theta = 0.6469$$

Second-score auction:

price: $p^{II}(\theta) = 2\theta$

quality: $q^{II}(\theta) = \sqrt{2\theta}$

score: $s^{II}(\theta) = \frac{q^{II}(\theta)}{p^{II}(\theta)} = \frac{1}{\sqrt{2\theta}}$

Expected score:

$$\Sigma^{II} = \int_{1.2}^{1.203731} s^{II}(\theta) f_2(\theta) d\theta = 0.6449$$

The above examples clearly demonstrate that the distribution of types plays a major role in the ranking of expected scores.

Even if the scoring rule and cost functions are the same, the ranking of expected revenues can get reversed if the distribution of types are different.

Hence, we need to put restrictions on both the scoring rule and the distribution function to get a ranking of expected scores.

Optimal Scoring auctions
(Che, Rand, 1993)

By the revelation principle any optimal outcome can be seen as a direct revelation mechanism.

Proposition:

In the optimal revelation mechanism, the firm with the lowest θ is selected; the winning firm is induced to choose quality q_0, which for each θ maximises

$$V(q) - c(q,\theta) - \frac{F(\theta)}{f(\theta)} c_\theta(q,\theta).$$

- In the optimal mechanism quality is distorted downwards to limit the information rents accruing to relatively efficient firms, while competition curtails the absolute magnitude of these rents.

- Compared to the optimal mechanism, the **naive** scoring rule (where $S(q,p) = V(q) - p$) entails excessive quality under first and second scoring auctions. It does so because it fails to take account of the information costs (the costs the buyer bears due to his inferior informational position) associated with increased quality and thus over-rewards quality.

- The above suggests that there is an incentive for the buyer to deviate from the naive scoring rule.

Consider the following scoring rule:

$$\tilde{S}(q,p) = V(q) - p - \Delta q$$

where

$$\Delta q = \int_{q_0(\bar{\theta})}^{q} \frac{F(q_0^{-1}(t))}{f(q_0^{-1}(t))} c_{q\theta}(t, q_0^{-1}(t))dt$$

for $q \in [q_0(\bar{\theta}), q_0(\underline{\theta})]$

and

$$\Delta q = \infty \text{ for } q \notin [q_0(\bar{\theta}), q_0(\underline{\theta})].$$

1. The rule differs from the true utility function (naive scoring rule) by the term Δq.

2. Roughly speaking, the rule subtracts additional points from a firm for an incremental increase in quality according to the function Δq.

Proposition:

(i) Under the scoring rule $\tilde{S}(q,p)$, the **first-score** and **second-score** auctions **implement** the optimal mechanism.

- With an appropriate scoring rule, the first-score and second-score auctions can implement the optimal outcome.

- As proposition 1 shows, an optimal mechanism induces a downward distortion of quality from the first best level to internalise the information costs of the buyer.

- This optimal downward distortion can be implemented by a scoring rule that penalises quality relative to the buyer’s actual valuation of quality.

1. Branco (RAND, 1997): Optimal mechanisms with one-dimensional quality and types (that are **correlated**).

2. Asker and Cantillon (RAND, 2010): Optimal mechanisms with one-dimensional quality and two-dimensional discrete types.

3. Nishimura (2015): Optimal mechanisms with multi-dimensional quality and single-dimensional types. Types are I.I.D.

Further research questions

1. For non-quasilinear scoring rules we concentrated mainly on single dimensional quality. Characterisation of equilibrium and ranking of expected scores when quality and types are multidimensional have not been analysed and is left for future research.

2. Optimal mechanisms (that maximise expected scores) have been derived in the literature for quasi-linear scoring rules (See Che, 1993, Asker Cantillon, 2010 and Nishimura, 2015).

3. However, such optimal mechanisms for general non-quasilinear scoring rules have not been analysed. This is an open question and is left for future research.

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